Remarks on an Interplay Between Algebra and PDE

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Dedicated to the memory of S. Shimorin, an extraordinary mathematician and a kind and gentle man.

Abstract We discuss O. Hesse's conjecture for homogeneous polynomials and B. I. Korenblum's conjecture on algebras of harmonic functions from the standpoint of nonlinear first-order PDE. Also, we extend a recent theorem of T. McKinley and B. Shekhtman for homogeneous polynomial partial differential operators to a wider class of linear PDE with entire coefficients.

1 Hesse's Conjecture

In 1859, O. Hesse [5] conjectured that if a homogenous polynomial u of $N > 1$ variables has a vanishing Hessian $\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)_{j,k=1}^{N}$, then the partial derivatives $\frac{\partial u}{\partial x_i}$, $j = 1,\dots,N$ are linearly dependent. In other words, Hess $u \equiv 0 \Leftrightarrow \nabla u := grad\, u : \mathbb{C}^N \to$ hyperplane. For example, let $N = 2$ and $u(x,y)$ is a homogeneous C^2-function of degree of homogeneity $k+1$, such that Hess $u = u_{xx}u_{yy} - u_{xy}^2 \equiv 0$. Let $u_x = f$, $u_y = g$, f,g are homogeneous of degree k. Then, $f_x g_y - f_y g_x = 0$ implies $\frac{f_x}{g_x} = \frac{f_y}{g_y} := \lambda$, while by homogeneity, $x f_x + y f_y = k f$ and $x g_x + y g_y = kg = \frac{1}{\lambda} x f_x + \frac{1}{\lambda} y f_y = \frac{k}{\lambda} f$. So, $f = \lambda g$ and $f_x = \lambda_x g + \lambda g_x$. Hence, $\lambda_x \equiv 0$ and, similarly, $\lambda_y \equiv 0$. Thus, $\lambda \equiv \text{const} = c$, $u_x = c u_y$ and ∇u maps $\mathbb{C}^2$ into a line.

P. Gordan and M. Nöther [2] showed that Hesse's conjuecture holds for $N = 2,3,4$ but is false for $n \geq 5$ in view of the following example of a cubic in 5 variables: $u(x_1,\dots,x_5) = x_1 x_4^4 + x_2 x_4 x_5 + x_3 x_5^2$. Indeed, denoting $D_j u = \frac{\partial u}{\partial x_j}$, we have

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$(D_1u)(D_3u) - (D_2u)^2 \equiv 0$. Hence, the components of ∇u are algebraically dependent, so Hess $u \equiv 0$ and $\nabla u : \mathbb{C}^5 \to \{x_1x_3 - x_2^2 = 0\}$ – cf. [10] for further discussion.

Note: u also satisfies a linear PDE, $D_1D_3\,u - D_2^2\,u = 0$. In other words, if we denote by $P(x_1,x_2,x_3) = x_1x_3 - x_2^2$, a quadratic homogeneous polynomial, then Gordan–Nöther quintick satisfies two equations: a nonlinear one, $P(\nabla u) = 0$; and a linear one, $P(D)(u) = 0$. We shall return to this point later in the discussion — cf. [10].

2 The Higher Ground: General Nonlinear First-Order PDE

Looking for the higher ground, one might ask whether if u, a holomorphic function, satisfies a "purely" nonlinear equation $F(\nabla u) = 0$, with $F : \mathbb{C}^N \to \mathbb{C}$ being an entire, or a meromorphic function with no linear factors, then the choices for u to be a global solution of such nonlinear equation are severely limited — e.g., perhaps forcing u to be linear. The Gordan–Nöther example, though crashing such hopes in general, is not overly satisfying since their u is a function of 5 variables while $F := x_1x_3 - x_2^2$ is a function of only 3 variables so F vanishes on the 2-dimensional linear subspace $\{(x_4,x_5)\}$. The following result is relevant to our discussion.

Theorem 1 (DK – [7]). *If an entire function u solves the (eiconal) equation $u_x^2 + y_y^2 - 1 = 0$, then u is linear.*

The proof was based on some elementary trick, thus missing the "correct," much more general, theorem.

Theorem 2 (J. Hemmati (Guerra) – [4]). *If $F : \mathbb{C}^2 \to \mathbb{C}$ is a meromorphic, purely nonlinear (cf. above) function and u is a meromorphic in $\mathbb{C}^2$ solution of $F(u_x, u_y) = 0$, then u is a linear function.*

Thus, in particular, if for a meromorphic in $\mathbb{C}^2$ function u, the gradient map, grad $u : \mathbb{C}^2 \to V$, maps $\mathbb{C}^2$ into an algebraic, nonlinear and irreducible variety V, u is a linear function and the grad u is a constant map – cf. [4]. This is, of course, a far-reaching generalization of Hesse's conjecture for $N = 2$. We refer to the survey [6] for recent extensions and generalizations of the Hemmati (Guerra) theorem.

Remark 1.

(i) Not only is Theorem 2 more general than its predecessor, Theorem 1, but its proof is much shorter and more to the point. Namely, it is easy to check [4] that the characteristics for $F(u_x, u_y)$ are all straight lines. Also, u_x, u_y stay constant on characteristics while nonlinearity implies that these characteristic lines have different slopes. This yields multivaluedness of u_x, u_y at the intersection points, thus implying that those functions have branching singularities and, hence, cannot be meromorphic.

(ii) Also, as another illustration of the failure of Hesse's conjecture in higher dimensions, Theorem 1 already fails in $\mathbb{C}^3$. The function $z-\varphi(x+iy):=u(x,y,z)$ satisfies the eiconal in $\mathbb{C}^3$ for any entire function φ of one variable. Moreover, in higher dimensions there are more and more opportunities for entire solutions of the eiconal $\sum_1^N \left(\frac{\partial u}{\partial z_j}\right)^2 = 1$. Take in $\mathbb{C}^5$, for example, $u = \varphi(z_1+iz_2)+\psi(z_3+iz_u)+z_5$ with entire φ,ψ, etc.

(iii) It is worth noticing that nonlinear equations in $\mathbb{R}^N$ are even more rigid. For example, as is well-known (cf. the references in [4, 7]), any C^1 solution u in $\mathbb{R}^N$ of $\sum_1^N u_{x_j}^2 = 1$ that is real-valued is linear. Indeed, the eiconal equation describes the velocity of light moving along the normals to the level surface with the constant speed ($=1$). If the level surfaces of u have nontrivial curvatures, the normals will intersect causing for the solution to become multivalued.

3 B. I. Korenblum's Conjecture

What happens when a solution of a linear PDE generates an algebra of solutions? Consider the following example.

Example 1. Let $P=\sum_1^N x_j^2$, $x_j\in\mathbb{R}$, so $P(D)=\Delta$. If $\Delta u=0$, and $\Delta u^2=0$, then $\Delta u^2 = 2u\Delta u + 2\sum_1^N \left(\frac{\partial u}{\partial x_j}\right)^2 = 0$, thus implying that u also satisfies a nonlinear equation $(\operatorname{grad} u)^2=0$, a similar equation to the eiconal. In the latter case, one can easily check that for all $k\in\mathbb{N}$, $\Delta\left(u^k\right)=0$, thus u generates an algebra of harmonic functions. For example, $\Delta u^3 = u\Delta u^2 + u^2\Delta u + 2u(\operatorname{grad} u)^2 = 0$, etc. In two variables, $\sum_1^2 \left(\frac{\partial u}{\partial x_j}\right)^2 = 0$ is equivalent to either $\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} = 0$, or $\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2} = 0$, thus making u either a holomorphic, or an anti-holomorphic function.

B. I. Korenblum [9] in the late 1970s conjectured that if $u\in C^2(\Omega)$, $\Omega\subset\mathbb{R}^3$ is a domain, and $\Delta u = \Delta u^2 = 0$ (and then $\Delta u^k = 0$, $k\in\mathbb{N}$), then, after an appropriate rotation of coordinates, u must be either an analytic or an anti-analytic function in two dimensions. Korenblum announced several proofs of the conjecture, all of which contained gaps.

The reason was that, as stated, the conjecture is false and the intensely developing theory of harmonic morphisms (cf., e. g., [1]) provides many counterexamples.

However, if we consider a global version of the conjecture, it might as well be true.

The following unpublished result by the author verifies the conjecture in the category of polynomials.

Theorem 3 (DK, '92, unpublished). *If u is a polynomial in $\mathbb{R}^3$ and $\Delta u = \Delta u^2 = 0$, then after an appropriate rotation of the coordinates, u must become an analytic or an anti-analytic function in 2 dimensions.*

The proof rests on the Lemma (DK, '92, unpublished), characterizing carriers of singularities of harmonic functions in $\mathbb{C}^3$.

Lemma 1 ([8, Prop. 20.1]). *Let $\varphi = \varphi(z_1, z_2, z_3)$ be a homogeneous polynomial of degree m such that the variety $\Gamma := \{z \in \mathbb{C}^3 : \varphi(z) = 0\}$ is everywhere characteristic (cf., e.g., [8, pp. 16, 53, 151] with respect to $\Delta := \sum_1^3 \frac{\partial^2}{\partial z_i^2}$, i. e., $\sum_1^3 \left(\frac{\partial \varphi}{\partial z_i}\right)^2 = 0$ on Γ. Then, up to a constant factor, either $\varphi(z_1, z_2, z_3) = \left(\sum_1^3 \alpha_j z_j\right)^m$, where $\alpha_j \in \mathbb{C}$ are constants such that $\sum_1^3 \alpha_j^2 = 0$, i.e., Γ is a characteristic (w.r.t. Δ) plane, or $\varphi(z_1, z_2, z_3) = \left(\sum_1^3 z_j^2\right)^{\frac{m}{2}}$, and Γ is an isotropic cone.*

For our purposes, we need the following obvious corollary.

Corollary 1. *If $\varphi(z_1, z_2, z_3)$ is a homogeneous polynomial of degree m satisfying an "eiconal" equation $\sum_1^3 \left(\frac{\partial \varphi}{\partial z_i}\right)^2 = 0$ in $\mathbb{C}^3$, then, up to a constant factor, $\varphi = \left(\sum_1^3 \alpha_j z_j\right)^m$, $\sum_1^3 \alpha_1^2 = 0$.*

We shall sketch the proof of the lemma later. Now, let us finish the proof of the theorem.

Let $u = u_0 + \cdots + u_m$, where u_j are homogeneous harmonic polynomials of degree $j \le m$. Then, clearly, the senior term u_m satisfies $\Delta u_m^2 = 0$, hence $\sum_1^3 \left(\frac{\partial u_m}{\partial z_i}\right)^2 \equiv 0$ and, by Corollary 1, $u_m = \left(\sum_1^3 \alpha_j z_i\right)^m$, with $\sum_1^3 \alpha_j^2 = 0$. Rotating the coordinate system in $\mathbb{C}^3$ we can assume without loss of generality that $u_m = c_m (z_1 + i z_2)^m$, where c is a constant. Now $u_{m-1} u_m$ is harmonic as well as the second senior term in the expansion of u^2 and since u_m^2 is harmonic. Therefore, $0 = \Delta(u_{m-1} u_m) = u_{m-1} \Delta u_m + u_m \Delta u_{m-1} + 2 \operatorname{grad} u_{m-1} \cdot c_m (1, i) (z_1 + i z_2)^{m-1} = 2C_m \left(\frac{\partial u_{m-1}}{\partial z_1} + i \frac{\partial u_{m-1}}{\partial z_2}\right)(z_1 + i z_2)^{m-1}$. Hence, $\frac{\partial u_{m-1}}{\partial z_1} + i \frac{\partial u_{m-1}}{\partial z_2} = 0$, yielding $u_{m-1} = c_{m-1}(z_1 + i z_2)^{m-1} + b z_3^{m-1}$. But $\Delta u_{m-1} = b(m-1)(m-2) z_3^{m-3} = 0$, yielding $b = 0$ and $u_{m-1} = c_{m-1}(z_1 + i z_2)^{m-1}$. Continuing this "backward" induction, we conclude that $u = P(z_1 + i z_2)$, where $P(u)$ is a polynomial of degree m of one variable. Thus, it remains to indicate the proof of the lemma.

Here are the main steps — cf. [8, Ch. 20, Sec. 2].

1. Solving $\varphi(z) = 0$ for one of the variables, say z_3, we obtain on $\Gamma = \{\varphi(z) = 0\}$, $z_3 = \psi(z_1, z_2)$, whose ψ, as is easily-verified, satisfies an eiconal equation

$\left(\psi_j := \frac{\partial \varphi}{\partial z_j}, j = 1,2\right)$, $\psi_1^2 + \psi_2^2 = -1$. φ is homogeneous of order m, so $\sum_1^3 z_j \varphi_j = m\varphi$, and the implicit differentiation yields $\psi_j = -\frac{\varphi_j}{\varphi_3}$, $j = 1,2$, so $-z_1\varphi_3\psi_1 - z_2\varphi_3\psi_2 + z_3\varphi_3 = m\varphi = 0$ on Γ.

2. Substituting $z_3 = \phi(z_1, z_2)$, we conclude that $z_1\psi_1 + z_2\psi_2 = \psi$, i.e., ϕ is homogeneous of order 1 function in 2 variables. Switching to polar coordinates r, θ we can write $\phi = rf(\theta)$ and it is easy to check that f satisfies an ODE $(f')^2 + f^2 = 1$. Differentiating the latter equation we obtain a second-order ODE that factors easily producing two solutions: (I) $f = \pm 1$, in which case Γ is an isotropic cone $\left\{\sum_1^3 z_j^2 = 0\right\}$, or (II) $f = \beta_1 \cos\theta + \beta_2 \sin\theta$, $\beta_1^2 + \beta_2^2 = 1$, in which case Γ is a plane.

Remarks:

(i) In view of the results on global solutions of the eiconal equations in 2D described in Section 2, Korenblum's conjecture holds for entire functions u in $\mathbb{C}^3$ as well. Indeed, as before, $\Delta u^2 = 0 \Rightarrow \sum_1^3 \left(\frac{\partial u}{\partial z_i}\right)^2 \equiv 0$, so on a level surface $\{u = c\}$, writing $z_3 = \psi(z_1, z_2)$, we have $(\psi_{z_1})^2 + (\psi_{z_2})^2 = -1$, i.e., ψ is a "global" solution of an eiconal, and hence must be linear. Therefore, all level surfaces of u are planes, and after a rotation, we conclude that $u = f(z_1 \pm i z_2)$, where f is an entire function of one variable.

(ii) With appropriate modifications one can show that an extended Korenblum's conjecture holds for polynomials in N variables but the statement must be adjusted, and loses is esthetic appeal. For example, in $\mathbb{C}^4$, $u = f(z_1 + iz_2) + g(z_3 - iz_4)$, where f, g are analytic functions of one variable, satisfy $\Delta u = \Delta u^2 = \cdots = \Delta u^k = \cdots = 0$. In higher dimensions there are even more opportunities to group the variables according to the same principle by taking corresponding vectors $(0, \ldots, \alpha_1, 0, \ldots, 0, \alpha_k, 0, \ldots)$, $k \le N$, $\sum_1^k \alpha_1^2 = 0$ in the isotropic cone $\Gamma_0 = \left\{z : \sum_1^N z_j^2 = 0\right\}$ and applying functions of one variable to dot products of these vectors with $z = (z_1, \ldots, z_N)$. We leave it to the interested reader to draw out the corresponding statements.

4 The T. McKinley–B. Shekhtman Conjecture

Recall that the Gordan–Nöther homogeneous cubic u in Section 1 satisfies two homogeneous equations: $P(\text{grad}\, u) = 0$ (first-order nonlinear equation), where $P(z_1, \ldots, z_5) = z_1 z_3 - z_2^2$, and a linear equation $P(D)\, u = D_1 D_3 u - D_2^2 u = 0$. In a recent elegant paper [10], T. McKinley and B. Shekhtman suggested that this is part of a general phenomenon.

Conjecture 1 (McKinley–Shekhtman, 2017). Let P, u be homogeneous polynomials. If $P(\operatorname{grad} u) = 0$, then $P(D)\,u = 0$.

The conjecture is based on the general feeling, underscored in Section 2, that global solutions of the first-order nonlinear PDE are quite special and scarce.

Example 2. As was shown in Section 3, a homogeneous polynomial u in $\mathbb{C}^3$ satisfying an "eiconal" $\sum_1^3 \left(\frac{\partial u}{\partial z_i}\right)^2 \equiv 0$, has a very special form $c\left(\sum_1^3 \alpha_j z_i\right)^m$, $\sum_1^3 \alpha_j^2 = 0$, thus obviously satisfying $\Delta u = 0$. We refer the reader to [10], where several special cases of the above conjecture are verified.

Also in [10], the following weak converse to the M–S conjecture is proved.

Theorem 4 ([10]). *Let P be a homogeneous polynomial while u is a polynomial. If $P(D)\left[f^k\right] \equiv 0$ for all $k \in \mathbb{N}$, then $P(\operatorname{grad} f) \equiv 0$.*

The proof in [10] is based on clever algebraic manipulations. Theorem 4 unexpectedly has a nice implication in the approximation theory based on the following result by A. Pinkus and B. Wajnryb [11].

Theorem 5 ([11]). *Let $f \in \mathbb{C}[z_1, \ldots, z_n)$ be a polynomial, then the following are equivalent:*

(i) $\mathscr{P}(f) := \operatorname{Span}\left\{[f(\cdot + b)]^k : b \in \mathbb{C}^N, k \in \mathbb{N}\right\} \neq \mathbb{C}[z_1, \ldots, z_N]$.
(ii) *$\exists$ polynomial $P : P(D)\left[f^k\right] = 0$, for all $k \in \mathbb{N}$*
(iii) *$\overline{\mathscr{P}(f)} \neq C\left(\mathbb{C}^N\right)$ with respect to the usual topology of convergence on compact subsets of $\mathbb{C}^N$. Invoking this, a nice corollary to Theorem 4 is given in [10].*

Invoking this, a nice corollary to Theorem 4 is given in [10].

Corollary 2. *Let f be a homogeneous polynomial. If $P(f) \neq C[z_1, \ldots, z_N]$, then there exists a homogeneous polynomial $P : P(\operatorname{grad} f) \equiv 0$, i.e., $\operatorname{grad} f : \mathbb{C}^N \to \mathbb{C}^N$ maps $\mathbb{C}^N$ into an algebraic variety and, hence, $\operatorname{Hess} f \equiv 0$.*

For the proof one just takes a senior homogeneous part of a polynomial guaranteed by Theorem 5 and applies Theorem 4. However, applying the standard classical result in PDE known as the Delassus–LeRoux theorem (cf. [8, pp. 22, 153], one can substantially expand Theorem 4 in [10] and the proof becomes much more straightforward and transparent.

Theorem 6. *Let $P(D) := \sum_{|\alpha| \le m} a_\alpha(z) D^\alpha$, $z = (z_1, \ldots, z_N)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_j \in \mathbb{N} \cup \{0\}$, $D^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_N}\right)^{\alpha_N}$ be a linear differential operator with entire coefficients a_α. Let $u : \mathbb{C}^N \to \mathbb{C}$ be an entire function and $P(D)\left[u^k\right] = 0$, for all k in some arithmetic progression (e.g., $k \in \mathbb{N}$, or $k = 2n+1$, $n \in \mathbb{N}$, etc.). Then, $\sum_{|\alpha|=m} a_\alpha(z)(\operatorname{grad} u)^\alpha = \sum_{|\alpha|=m} a_\alpha(z)\left(\frac{\partial u}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial u}{\partial z_N}\right)^{\alpha_N} \equiv 0$. Thus, $\operatorname{grad} u$ maps $\mathbb{C}^N$ into an analytic hypersurface and, hence, $\operatorname{Hess} u \equiv 0$.*

(Theorem 4 follows at once from Theorem 6 when $P = \sum_{|\alpha|=m} a_\alpha z^\alpha$, a homogeneous polynomial, i.e., $P(D)$ is a constant coefficients operator.)

The following result of Delassus–LeRoux is the key.

Lemma 2 (cf. [8, pp. 22, 153 and the references there]). *Let $\Gamma := \{z : \varphi(z) = 0\}$ be a non-singular analytic hypersurface in $\mathbb{C}^N$ and v be a holomorphic solution of $P(D)\, v = 0$ in $\mathbb{C}^N \smallsetminus \Gamma$ and v is singular everywhere on Γ. Then, Γ is everywhere characteristic with respect to $P(D)$, i.e., $\sum_{|\alpha|=m} a_\alpha(z)(\operatorname{grad} \varphi)^\alpha \equiv 0$ on Γ.*

Remark 2. The Delassus–LeRoy theorem says simply that the singularities of solutions of linear analytic PDE "propagate" through $\mathbb{C}^N$ exclusively along characteristic surfaces.

From Lemma 2, Theorem 6 follows almost at once.

Proof. First assume, for the sake of clarity, $P(D)\left[u^k\right] = 0$, for all $k \in N$. For any $c \in C$, in an open neighborhood where $|u| < |c|$ we have $f := \frac{1}{c-u} = 1/c \sum_0^\infty \frac{u^k}{c^k}$, and the series converges. Hence, by the hypothesis, $P(D)(f) = 0$ in that neighborhood, and by analytic continuation everywhere in $\mathbb{C}^N \smallsetminus \{u = c\}$. By Lemma 2, $\Gamma_c := \{u = c\}$ must be everywhere characteristic with respect to $P(D)$, i.e., $\sum_{|\alpha|=m} a_\alpha(z)(\operatorname{grad} u)^\alpha \equiv 0$ on Γ_c. But taking a continual family of Γ_c, c runs over an open set in $\mathbb{C}$, we arrive at the conclusion of the theorem.

The proof is easily modified to establish the theorem in full generality. Indeed, if the hypothesis holds for $k = n\ell + d$, $\ell, d \in \mathbb{N}$, fixed, $n = 1, 2, \ldots$ we can always write

$$f_c := \frac{u^d}{c^\ell - u^\ell} = \sum_{n=0}^\infty \frac{u^d}{c^\ell} \left(\frac{u}{c}\right)^{n\ell} = \sum_{r=0}^\infty \frac{u^{n\ell+d}}{c^{(n+n\ell)}}$$

and then proceed exactly as before.

Remarks:

(i) The classical ("calculus") proof of the Delassus–LeRoux theorem can be found in [3, Ch. 3]. A modern proof based on the elementary but far-reaching extension of the Cauchy–Kovalevskaya theorem due to M. Zerner (1971) is in [8, pp. 22, 153].

(ii) Instead of the family of functions $\left\{\frac{1}{u-c}, c \in \mathbb{C}\right\}$, one can, of course, take dilations of any function $f(u)$ with finitely many singularities on the circle of convergence of its Taylor series. We leave the straightforward details of formulating the corresponding result to the reader.

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